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Radiation of sound waves from a rigid stepped cylindrical waveguide

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Abstract. The radiation of plane harmonic sound waves from a rigid stepped cylindrical waveguide is treated by using the mode-matching method in conjunction with theWiener-Hopf technique. The solution is exact, but formal, since infinite series of unknowns and some branch-cut integrals with unknown integrands are involved. Approximation procedures based on rigorous asymptotics are used and the approximate solution to the Wiener-Hopf equations is derived in terms of infinite series of unknowns, which are determined from infinite systems of linear algebraic equations. Numerical solutions of these systems are obtained for various values of the parameters of the problem and their effects on the directivity of the stepped waveguide is presented.

Key words: acoustic radiation, circular waveguide, integral equations, step discontinuity, Wiener-Hopf Technique

1. Introduction

Different types of acoustic or electromagnetic stepped waveguides are commonly used as loudspeakers or as primary feeds in reflector-antenna systems used in microwave communications. To analyze the performance of such radiators, one needs to know accurately their near- and far-field patterns.

The radiation characteristics of circular waveguides have been the subject of numerous past investigations. The first rigorous analytical solution of the radiation from a semi-infinite, infinitely thin unflanged circular rigid pipe, was obtained by Levine and Schwinger [1]. Later Ando [2] considered the same problem, in the case of non-vanishing wall thickness. Rawlins [3] investigated the radiation of sound from a rigid cylindrical duct with an acoustically absorbing internal surface. The analysis reported in [2] was recently generalized by Büyükaksoy and Polat [4] to the case where the inner and outer surfaces of the pipe are impedance boundaries.

Notice that the acoustic properties of slowly varying and stepped cylindrical ducts have been tackled by Nayfeh and Telionis [6], Rienstra [6] and Nilsson and Brander [7]. In [7], the reflection and transmission of sound in a cylindrical waveguide with a jump in its diameter is treated through the Wiener-Hopf technique.

The aim of this work is to study the pressure directivity of an open-ended stepped circular pipe asymptotically. To this end we consider the problem of plane harmonic sound waves propagating out of a semi-infinite duct, via an other coaxial cylindrical duct of finite length and bigger radius and then issuing into free space. Note that a similar electromagnetic radiation problem has been considered by Birbir, Büyükaksoy and Chumachenko [8] for the case of a two-dimensional box-like horn. The method adopted here is similar to that employed in [8] and consists of expressing the total field in the waveguide region in terms of normal waveguide

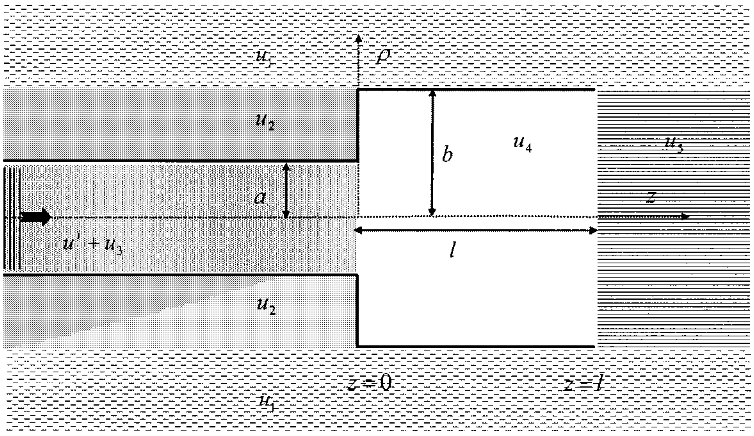

Figure 1. Stepped cylindrical waveguide

modes and using the Fourier Transform elsewhere. Then, the related boundary-value problem is formulated as a Modified Wiener-Hopf Equation of the third kind and then reduced to a pair of simultaneous Fredholm integral equations of the second kind which are susceptible to a treatment by iterations. The formal solution involves branch-cut integrals with unknown integrands and infinitely many unknown expansion coefficients satisfying infinite systems of linear algebraic equations. The branch-cut integrals are evaluated asymptotically for large values of the acoustical length of the finite duct and the linear systems of algebraic equations are solved numerically for various values of the parameters of the problem, such as the radii of the ducts and the acoustical length of the finite waveguide and their effects on the radiation phenomenon are shown graphically. The results are found to be in good agreement with the experimental ones related to the circular-pipe horn loudspeaker reported by Ando [9]

Notice also that the present acoustic radiation problem can also be considered as a good starting point for the analysis of the corresponding electromagnetic problem, that is the circular cylindrical-horn antenna.

The time dependence is assumed to be $\exp(-\mathrm{i}\omega t)$, with ω being the angular frequency, and is suppressed throughout the paper.

2. Analysis

Consider the radiation of a time-harmonic plane sound wave propagating along the positive z direction from an acoustically rigid cylindrical horn defined by $\{\rho = a\ ,\ z \in (-\infty, 0)\} \cup \{\rho \in (a, b)\ ,\ z = 0\} \cup \{\rho = b\ ,\ z \in (0, l)\}$, where (ρ, ϕ, z) denote the usual cylindrical polar coordinates (see Figure 1).

From the symmetry of the geometry of the problem and of the incident field, the acoustic field will be independent of ϕ everywhere. We shall therefore introduce a scalar potential $u(\rho, z)$ which defines the acoustic pressure and velocity by $p = \mathrm{i}\omega\rho_0 u$ and $\mathbf{v} =$ grad u, respectively, where ρ_0 is the density of the undisturbed medium.

Let the incident field be given by

$$u^{\mathrm{i}} = \exp(\mathrm{i}kz)\,, \tag{1}$$

where $k = \omega/c$ denotes the wave number. For the sake of analytical convenience we will assume that the surrounding medium is slightly lossy and k has a small positive imaginary part. The lossless case can be obtained by letting $\Im\mathfrak{m}k \to 0$ at the end of the analysis.

The total field $u^T(\rho, z)$ can be written as

$$u^T(\rho, z) = \begin{cases} u_1(\rho, z); & \rho > b, z \in (-\infty, \infty) \\ u_2(\rho, z); & \rho \in (a, b), z < 0 \\ u_3(\rho, z) + u^i(\rho, z); & \rho \in (0, a), z < 0 \\ u_4(\rho, z); & \rho \in (0, b), z \in (0, l) \\ u_5(\rho, z); & \rho \in (0, b), \ z > l, \end{cases} \tag{2}$$

where u^i is the incident field as given by (1) and $u_j(\rho, z),\ j = 1-5$, denote the scattered fields $u_j(\rho, z),\ j = 1-5$, which satisfy the Helmholtz equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right]u_j(\rho, z) = 0, \qquad j = 1, 2, 3, 4, 5, \tag{3}$$

is to be determined with the help of the following boundary and continuity relations:

$$\frac{\partial}{\partial\rho}u_1(b, z) = 0, \ \ z\epsilon(0, l), \qquad \frac{\partial}{\partial\rho}u_2(a, z) = 0, \ \ z < 0, \tag{4a,b}$$

$$\frac{\partial}{\partial\rho}u_3(a, z) = 0, \ \ z < 0, \qquad \frac{\partial}{\partial\rho}u_4(b, z) = 0; \ \ z \in (0, l), \tag{4c,d}$$

$$\frac{\partial u_2}{\partial z}(\rho, 0) = 0; \ \ \rho \in (a, b), \qquad \frac{\partial u_4}{\partial z}(\rho, 0) = 0; \ \ \rho \in (a, b), \tag{4e,f}$$

$$u_1(b, z) = u_2(b, z); \quad z < 0, \qquad \frac{\partial u_1}{\partial\rho}(b, z) = \frac{\partial u_2}{\partial\rho}(b, z); \quad z < 0, \tag{4g,h}$$

$$u_1(b, z) = u_5(b, z); \quad z > l, \qquad \frac{\partial u_1}{\partial\rho}(b, z) = \frac{\partial u_5}{\partial\rho}(b, z); \quad z > l, \tag{4i,j}$$

$$u_3(\rho, 0) + u^i(0) = u_4(\rho, 0); \quad \rho \in (0, a), \tag{4k}$$

$$\frac{\partial}{\partial z}u_3(\rho, 0) + \frac{\partial}{\partial z}u^i(0) = \frac{\partial}{\partial z}u_4(\rho, 0); \quad \rho \in (0, a), \tag{4l}$$

$$u_4(\rho, l) = u_5(\rho, l); \quad \rho \in (0, b), \qquad \frac{\partial u_4}{\partial z}(\rho, l) = \frac{\partial u_5}{\partial z}(\rho, l); \quad \rho \in (0, b). \tag{4m,n}$$

To ensure the uniqueness of the mixed boundary-value problem stated by (3) and (4a–n), one has to take into account the following radiation and edge conditions:

$$u \sim \frac{e^{ikr}}{r}, \quad r = \sqrt{\rho^2 + z^2} \to \infty, \tag{5}$$

$$u^T(b + 0, z) = O(1), \quad z \to -0, \tag{6a}$$

$$\frac{\partial}{\partial\rho}u^T(b+0,z) = O(z^{-1/3}), \quad z \to -0, \tag{6b}$$

$$u^T(b,z) = O(1), \quad z \to l+0, \tag{6c}$$

$$\frac{\partial}{\partial\rho}u^T(b,z) = O((z-l)^{-1/2}), \quad z \to l+0. \tag{6d}$$

2.1. Reduction to a Modified Wiener-Hopf Equation

In the region $\rho > b$, the scattered field $u_1(\rho, z)$ satisfies the Helmholtz equation for $z \in (-\infty, \infty)$. Multiplying (3) by $e^{i\alpha z}$ with α being the Fourier-transform variable and integrating the resultant equation with respect to z from $-\infty$ to ∞, we obtain

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + (k^2-\alpha^2)\right]F(\rho,\alpha) = 0, \tag{7a}$$

where

$$F(\rho,\alpha) = F_-(\rho,\alpha) + F_1(\rho,\alpha) + e^{i\alpha l}F_+(\rho,\alpha). \tag{7b}$$

$F_+(\rho,\alpha)$, $F_-(\rho,\alpha)$ and $F_1(\rho,\alpha)$ are defined by

$$F_-(\rho,\alpha) = \int_{-\infty}^{0} u_1(\rho,z)e^{i\alpha z}dz, \quad F_+(\rho,\alpha) = \int_{l}^{\infty} u_1(\rho,z)e^{i\alpha(z-l)}dz, \tag{7c,d}$$

and

$$F_1(\rho,\alpha) = \int_0^l u_1(\rho,z)e^{i\alpha z}dz, \tag{7e}$$

respectively. Owing to the analytical properties of Fourier integrals, $F_+(\rho,\alpha)$ and $F_-(\rho,\alpha)$ are regular functions in the upper half-plane $\mathfrak{Im}\alpha > \mathfrak{Im}(-k)$ and in the lower half-plane $\mathfrak{Im}\alpha < \mathfrak{Im}k$, respectively, while $F_1(\rho,\alpha)$ is an entire function.

The solution of (7a) satisfying the radiation condition for $\rho > b$ reads

$$F_-(\rho,\alpha) + F_1(\rho,\alpha) + e^{i\alpha l}F_+(\rho,\alpha) = -A(\alpha)\frac{H_0^{(1)}(K\rho)}{K(\alpha)H_1^{(1)}(Kb)} \tag{8a}$$

with

$$K(\alpha) = \sqrt{k^2-\alpha^2}. \tag{8b}$$

In (8a), $H_n^{(1)}$ stands for the Hankel function of the first kind and n-th order, given by

$$H_n^{(1)} = J_n + iY_n \tag{8c}$$

while $A(\alpha)$ is a spectral coefficient to be determined. The square-root function $K(\alpha)$ is defined in the complex α-plane cut as shown in Figure 2 such that $K(0) = k$.

Consider now the Fourier transform of (4a), namely

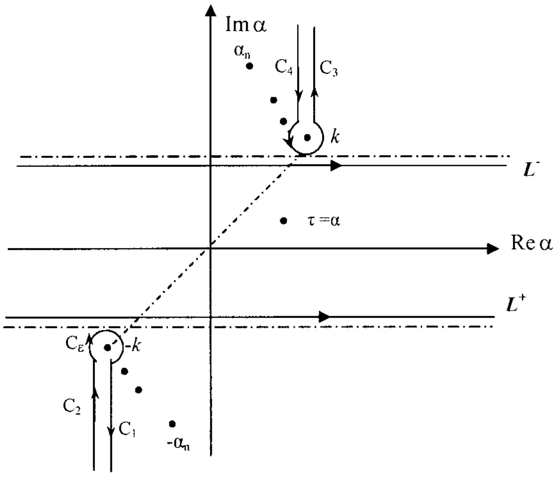

Figure 2. Branch-cuts and integration lines in the complex plane

$$\dot{F}_1(b,\alpha)=0 \tag{9}$$

where the dot specifies the derivative with respect to ρ. The differentiation of (8a) with respect to ρ yields

$$\dot{F}_-(\rho,\alpha)+\dot{F}_1(\rho,\alpha)+\mathrm{e}^{\mathrm{i}\alpha l}\dot{F}_+(\rho,\alpha)=A(\alpha)\frac{H_1^{(1)}(K\rho)}{H_1^{(1)}(Kb)}. \tag{10}$$

Setting $\rho=b$ in (10) and using (9), we obtain

$$A(\alpha)=\dot{F}_-(b,\alpha)+\mathrm{e}^{\mathrm{i}\alpha l}\dot{F}_+(b,\alpha). \tag{11a}$$

Now the elimination of $A(\alpha)$ between (8a) and (11a) gives

$$F_-(b,\alpha)+\mathrm{e}^{\mathrm{i}\alpha l}F_+(b,\alpha)=-F_1(b,\alpha)-\frac{H_0^{(1)}(K\rho)}{KH_1^{(1)}(Kb)}\left[\dot{F}_-(b,\alpha)+\mathrm{e}^{\mathrm{i}\alpha l}\dot{F}_+(b,\alpha)\right]. \tag{11b}$$

In the regions $a<\rho<b$, $z<0$ and $0<\rho<b$, $z>l$, the scattered fields $u_2(\rho,z)$ and $u_5(\rho,z)$ satisfy the Helmholtz equation in (3) whose solutions read

$$\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho\frac{\mathrm{d}}{\mathrm{d}\rho})+K^2(\alpha)\right)G_-(\rho,\alpha)=\mathrm{i}\alpha f(\rho) \tag{12a}$$

and

$$\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho\frac{\mathrm{d}}{\mathrm{d}\rho})+K^2(\alpha)\right)G_+(\rho,\alpha)=g(\rho)-\mathrm{i}\alpha h(\rho), \tag{12b}$$

respectively, where the boundary condition (4e) satisfied on the rigid end has been taken into account. In (12a) and (12b), $G_+(\rho,\alpha)$ and $G_-(\rho,\alpha)$ are defined by:

$$G_+(\rho,\alpha)=\int_l^\infty u_5(\rho,z)\mathrm{e}^{\mathrm{i}\alpha(z-l)}\mathrm{d}z \tag{12c}$$

and

$$G_-(\rho,\alpha) = \int_{-\infty}^{0} u_2(\rho,z)e^{i\alpha z}dz, \tag{12d}$$

respectively, while $f(\rho)$, $g(\rho)$ and $h(\rho)$ stand for

$$f(\rho) = u_2(\rho,0), \quad g(\rho) = \frac{\partial}{\partial z}u_5(\rho,l), \quad h(\rho) = u_5(\rho,l). \tag{12e,f,g}$$

$G_+(\rho,\alpha)$ and $G_-(\rho,\alpha)$ are regular functions in the upper ($\mathfrak{Im}\alpha > \mathfrak{Im}(-k)$) and lower ($\mathfrak{Im}\alpha > \mathfrak{Im}(-k)$) halves of the α-plane.

Particular solutions to (12a) and (12b) can be found easily by using the Green's function technique. The Green's function related to (12a) satisfies the Helmholtz equation

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + K^2(\alpha)\right]\mathcal{G}_1(\rho,\alpha) = 0, \quad \rho \neq t\ , \quad \rho, t \in (a,b) \tag{13a}$$

with the following conditions:

$$\mathcal{G}_1(t+0,t,\alpha) = \mathcal{G}_1(t-0,t,\alpha), \tag{13b}$$

$$\frac{\partial}{\partial\rho}\mathcal{G}_1(t+0,t,\alpha) - \frac{\partial}{\partial\rho}\mathcal{G}_1(t-0,t,\alpha) = \frac{1}{t}, \tag{13c}$$

$$\frac{\partial}{\partial\rho}\mathcal{G}_1(b,t,\alpha) = 0, \quad \frac{\partial}{\partial\rho}\mathcal{G}_1(a,t,\alpha) = 0. \tag{13d,e}$$

The solution is

$$\mathcal{G}_1(\rho,t,\alpha) = \frac{1}{M(\alpha)}Q_1(\rho,t,\alpha) \tag{14a}$$

with

$$Q_1(\rho,t,\alpha) = \frac{\pi}{2}\begin{cases} [J_0(K\rho)Y_1(Ka) - J_1(Ka)Y_0(K\rho)] \\ \qquad \times [J_0(Kt)Y_1(Kb) - J_1(Kb)Y_0(Kt)]\,, a \le \rho \le t \\ [J_0(K\rho)Y_1(Kb) - J_1(Kb)Y_0(K\rho)] \\ \qquad \times [J_0(Kt)Y_1(Ka) - J_1(Ka)Y_0(Kt)]\ , t \le \rho \le b \end{cases} \tag{14b}$$

and

$$M(\alpha) = [J_1(Ka)Y_1(Kb) - J_1(Kb)Y_1(Ka)]\,. \tag{14c}$$

The solution of (12a) can now be written as

$$G_-(\rho,\alpha) = \frac{1}{M(\alpha)}\left\{\frac{D_1(\alpha)}{K(\alpha)}[J_0(K\rho)Y_1(Ka) - Y_0(K\rho)J_1(Ka)] + i\alpha\int_a^b f(t)Q_1(t,\rho,\alpha)t\,dt\right\}. \tag{15}$$

Similarly, to obtain the particular solution of (12b), we will use again the Green's function technique. The Green function satisfying the Helmholtz equation

$$\left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right)+K^2(\alpha)\right)\mathcal{G}_2\left(\rho,t,\alpha\right)=0,\,\rho\neq t,\,\rho,t\in(0,b) \tag{16a}$$

under the following conditions

$$\mathcal{G}_2(0,t,\alpha)\sim \text{ bounded},\quad \mathcal{G}_2(t+0,t,\alpha)=\mathcal{G}_2(t-0,t,\alpha), \tag{16b,c}$$

$$\frac{\partial}{\partial\rho}\mathcal{G}_2(t+0,t,\alpha)-\frac{\partial}{\partial\rho}\mathcal{G}_2(t-0,t,\alpha)=\frac{1}{t},\quad \frac{\partial}{\partial\rho}\mathcal{G}_2(b,t,\alpha)=0, \tag{16d,e}$$

is

$$\mathcal{G}_2\left(\rho,t,\alpha\right)=\frac{1}{J_1(Kb)}Q_2\left(\rho,t,\alpha\right) \tag{17a}$$

with

$$Q_2\left(\rho,t,\alpha\right)=\frac{\pi}{2}\begin{cases}J_0(K\rho)\left[J_1(Kb)Y_0(Kt)-J_0(Kt)Y_1(Kb)\right], & 0\leq\rho\leq t\\ J_0(Kt)\left[J_1(Kb)Y_0(K\rho)-J_0(K\rho)Y_1(Kb)\right], & t\leq\rho\leq b\end{cases}. \tag{17b}$$

Now, $G_+(\rho,\alpha)$ reads

$$G_+(\rho,\alpha)=\frac{1}{J_1(Kb)}\left\{\frac{D_2(\alpha)}{K(\alpha)}J_0(K\rho)+\int_0^b\left[g(t)-\mathrm{i}\alpha h(t)\right]Q_2(t,\rho,\alpha)t\mathrm{d}t\right\}. \tag{18}$$

In (15) and (18), $D_1(\alpha)$ and $D_2(\alpha)$ are spectral coefficients to be determined while f, g and h are given by (12d), (12e) and (12f), respectively. Differentiating (15) and (18) with respect to ρ, we obtain

$$\dot{G}_-(\rho,\alpha)=\frac{-1}{M(\alpha)}\left\{D_1(\alpha)\left[J_1(K\rho)Y_1(Ka)-Y_1(K\rho)J_1(Ka)\right]-\mathrm{i}\alpha\int_a^b f(t)\dot{Q}_1(t,\rho,\alpha)t\mathrm{d}t\right\} \tag{19a}$$

and

$$\dot{G}_+(\rho,\alpha)=\frac{-1}{J_1(Kb)}\left\{D_2(\alpha)J_1(K\rho)-\int_0^b\left[g(t)-\mathrm{i}\alpha h(t)\right]\dot{Q}_2(t,\rho,\alpha)t\mathrm{d}t\right\}. \tag{19b}$$

The continuity relations in (4b) and (4d) require

$$\dot{F}_-(b,\alpha)=\dot{G}_-(b,\alpha),\quad \dot{F}_+(b,\alpha)=\dot{G}_+(b,\alpha). \tag{20a,b}$$

Replacing $\dot{G}_+(b,\alpha)$ and $\dot{G}_-(b,\alpha)$ appearing in (20a) and (20b) by their expressions given in (19a) and (19b), respectively, we can solve $D_1(\alpha)$ and $D_2(\alpha)$ uniquely as

$$\dot{F}_-(b,\alpha)=D_1(\alpha),\quad \dot{F}_+(b,\alpha)=-D_2(\alpha). \tag{21a,b}$$

By substituting $D_1(\alpha)$ and $D_2(\alpha)$ given by (21a) and (21b) in (15) and (18) we get

$$G_-(\rho,\alpha)=\frac{1}{M(\alpha)}\left\{\frac{\dot{F}_-(b,\alpha)}{K(\alpha)}\left[J_0(K\rho)Y_1(Ka)-Y_0(K\rho)J_1(Ka)\right]+\mathrm{i}\alpha\int_a^b f(t)Q_1(t,\rho,\alpha)t\mathrm{d}t\right\} \tag{22a}$$

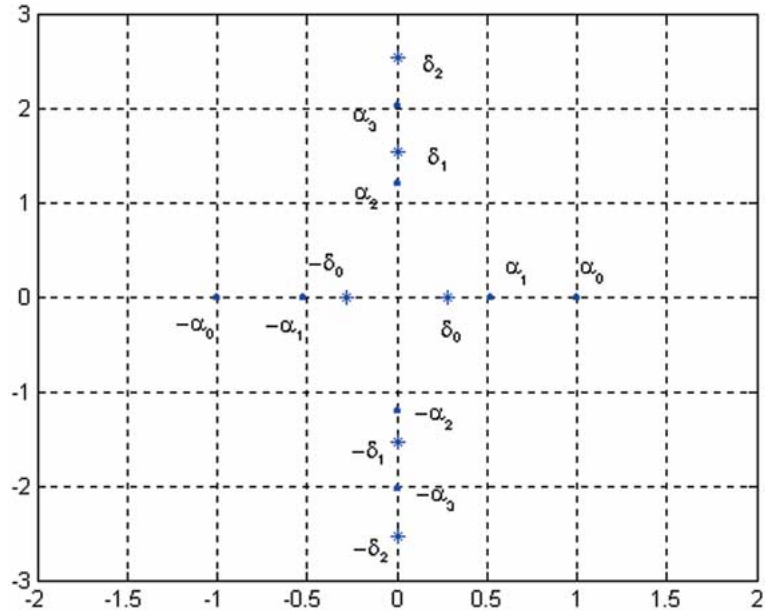

Figure 3. The location of the zeros α_m and δ_m in the complex plane for $k = 1$, $a = 1$ and $b = 4{\cdot}5$.

and

$$G_+(\rho,\alpha) = \frac{1}{J_1(Kb)}\left\{-\frac{\dot{F}_+(b,\alpha)}{K(\alpha)}J_0(K\rho) + \int_0^b \left[g(t) - \mathrm{i}\alpha h(t)\right] Q_2(t,\rho,\alpha)t\mathrm{d}t\right\}. \quad \text{(22b)}$$

The left-hand sides of (22a) and (22b) are regular in the half-planes $\mathfrak{Im}(\alpha) < \mathfrak{Im}(k)$ and $\mathfrak{Im}(\alpha) > \mathfrak{Im}(-k)$ respectively. By using the following relations

$$J_\nu\left(\mathrm{e}^{\mathrm{i}\pi}z\right) = \mathrm{e}^{\mathrm{i}\nu\pi}J_\nu(z), \quad Y_\nu\left(\mathrm{e}^{\mathrm{i}\pi}z\right) = \mathrm{e}^{-\mathrm{i}\nu\pi}Y_\nu(z) + 2\mathrm{i}\cos\,\nu\pi\, J_\nu(z) \quad \text{(22c)}$$

one can easily check that the right-hand sides of (22a) and (22b) are continuous across the branch-cuts lying in the lower and upper halves of the complex α-plane and consequently have no branch-points in their respective regions of analyticity. However, their regularity may be violated by the presence of simple poles lying in the lower and upper half planes, namely at $\alpha = -\delta_m$ ($\mathfrak{Im}\delta_m > \mathfrak{Im}k$) and $\alpha = \alpha_m$, ($\mathfrak{Im}\alpha_m > \mathfrak{Im}k$), respectively, with

$$J_1(Z_m a)Y_1(Z_m b) - J_1(Z_m b)Y_1(Z_m a) = 0, \quad Z_m = K(\delta_m), \quad m = 0, 1, 2, \ldots \quad \text{(23a)}$$

and

$$J_1(\xi_m) = 0, \quad \alpha_m = \sqrt{k^2 - (\xi_m/b)^2}, \quad m = 0, 1, 2, \ldots . \quad \text{(23b)}$$

Equations (22a) and (22b) are indeed regular at $\alpha = -\delta_m$ and $\alpha = \alpha_m$ in their respective regions of regularity, if we have

$$\begin{aligned}\dot{F}_-(b,-\delta_m) = {} & -\frac{\mathrm{i}\pi}{2}\delta_m Z_m \frac{J_1(Z_m b)}{J_1(Z_m a)}\int_a^b f(t)\left[J_1(Z_m a)Y_0(Z_m t)\right. \\ & \left. - J_0(Z_m t)Y_1(Z_m a)\right] t\mathrm{d}t, \quad m = 0, 1, 2, \ldots ,\end{aligned} \quad \text{(24)}$$

$$\dot{F}_{+}(b,\alpha_m) = -\frac{\pi}{2}\frac{\xi_m}{b}Y_1(\xi_m)\int\limits_0^b \left[g(t) - \mathrm{i}\alpha h(t)\right] J_0(\frac{\xi_m}{b}t)t\mathrm{d}t, \quad m \neq 0, \tag{25a}$$

$$\dot{F}_{+}(b,k) = \frac{1}{b}\int\limits_0^b \left[g(t) - \mathrm{i}kh(t)\right] t\mathrm{d}t, \quad m = 0. \tag{25b}$$

Using the continuity relations in (4g) and (4i),we write

$$F_{-}(b,\alpha) + \mathrm{e}^{\mathrm{i}\alpha l}F_{+}(b,\alpha) = G_{-}(b,\alpha) + \mathrm{e}^{\mathrm{i}\alpha l}G_{+}(b,\alpha). \tag{26}$$

Recalling (11b), we obtain

$$-\frac{b}{2}F_1(b,\alpha) + \frac{\dot{F}_{-}(b,\alpha)}{Q\,(\alpha)\,K^2(\alpha)} + \frac{\mathrm{e}^{\mathrm{i}\alpha l}\dot{F}_{+}(b,\alpha)}{K(\alpha)R\,(\alpha)} = -\frac{\mathrm{i}\alpha}{2K(\alpha)M(\alpha)}\int\limits_a^b f(t)\left[Y_1(Ka)J_0(Kt)\right.$$
$$\left.-J_1(Ka)Y_0(Kt)\right]t\mathrm{d}t + \frac{\mathrm{e}^{\mathrm{i}\alpha l}}{2K(\alpha)J_1(Kb)}\int\limits_0^b \left[g(t) - \mathrm{i}\alpha h(t)\right] J_0(Kt)t\mathrm{d}t \tag{27a}$$

with

$$R(\alpha) = \mathrm{i}\pi J_1(Kb)H_1^{(1)}(Kb) \quad \text{and} \quad Q(\alpha) = \frac{H_1^{(1)}(Ka)}{\pi H_1^{(1)}(Kb)M(\alpha)}. \tag{27b,c}$$

Since $f(t)$, $g(t)$ and $h(t)$ are absolutely integrable functions satisfying Dini conditions, they can be expanded into series of the following complete sets of orthogonal functions [10, p.453 and p.449].

$$f(t) = \sum_{m=0}^{\infty} f_m\left[J_1(Z_m a)Y_0(Z_m t) - Y_1(Z_m a)J_0(Z_m t)\right] \tag{28a}$$

and

$$g(t) = \sum_{m=0}^{\infty} g_m J_0(\frac{\xi_m}{b}t), \quad h(t) = \sum_{m=0}^{\infty} h_m J_0(\frac{\xi_m}{b}t), \tag{28b,c}$$

where f_m, g_m and h_m are related to $f(t)$, $g(t)$ and $h(t)$ through [10, p.453 and p.449]

$$f_m = \frac{\pi^2}{2}\frac{J_1^2(Z_m b)Z_m^2}{J_1^2(Z_m a) - J_1^2(Z_m b)}\int\limits_a^b f(t)\left[J_1(Z_m a)Y_0(Z_m t) - Y_1(Z_m a)J_0(Z_m t)\right] t\mathrm{d}t, \tag{29}$$

$$g_m = \frac{2}{b^2 J_0^2(\xi_m)}\int\limits_0^b g(t)J_0(\frac{\xi_m}{b}t)t\mathrm{d}t, \quad m \neq 0, \tag{30a}$$

$$g_0 = \frac{2}{b^2}\int\limits_0^b g(t)t\mathrm{d}t, \qquad m = 0, \tag{30b}$$

and

$$h_m = \frac{2}{b^2 J_0^2(\xi_m)} \int_0^b h(t) J_0(\frac{\xi_m}{b} t) t \mathrm{d}t, \quad m \neq 0, \tag{30c}$$

$$h_0 = \frac{2}{b^2} \int_0^b h(t) t \mathrm{d}t, \quad m = 0. \tag{30d}$$

By taking into account (23),(24a,b) and (29), and (30a-d), we can express f_m, g_m and h_m in terms of $\dot{F}_+(a, \alpha_m)$ and $\dot{F}_-(a, -\alpha_m)$ as follows:

$$f_m = \frac{\pi Z_m}{\mathrm{i}\delta_m} \frac{J_1(Z_m a) J_1(Z_m b)}{J_1^2(Z_m b) - J_1^2(Z_m a)} \dot{F}_-(b, -\delta_m) \tag{31a}$$

and

$$g_m - \mathrm{i}\alpha_m h_m = -\frac{2}{b} \frac{\dot{F}_+(b, \alpha_m)}{J_0(\xi_m)}, \quad g_0 - \mathrm{i}\alpha_0 h_0 = -\frac{2}{b} \dot{F}_+(b, k). \tag{31b,c}$$

By using the edge conditions, the asymptotic expressions of the Bessel's functions valid for large arguments and the following asymptotic estimates (see [11])

$$\delta_m = \frac{\mathrm{i}m\pi}{b-a} + \mathcal{O}\left(\frac{1}{m}\right), \quad \alpha_m = \frac{\mathrm{i}m\pi}{b} + \frac{\mathrm{i}\pi}{4b} + \mathcal{O}\left(\frac{1}{m}\right) \quad m \to \infty, \tag{31d}$$

we can show easily that

$$f_m = \mathcal{O}\left(\mathrm{e}^{-m\pi} \left[(b-a) / (m\pi)\right]^{2/3}\right) g_m / (\mathrm{i}\alpha_m) - h_m = \mathcal{O}\left(\mathrm{e}^{-m\pi/b} / (m\pi/b)\right), \quad m \to \infty. \tag{31e}$$

Substituting (28a-c) and (29), (30a-d) in (27a) and evaluating the resulting integrals, we obtains the following Modified Wiener-Hopf Equation (MWHE) of the third kind valid in the strip $\mathfrak{Im}(-k) < \mathfrak{Im}(\alpha) < \mathfrak{Im}(k)$

$$-\frac{b}{2} F_1(b, \alpha) + \frac{\dot{F}_-(b, \alpha)}{K^2(\alpha)} Q(\alpha) + \frac{\mathrm{e}^{\mathrm{i}\alpha l} \dot{F}_+(b, \alpha)}{K^2(\alpha) R(\alpha)} = \frac{\mathrm{i}\alpha}{\pi} \sum_{m=0}^{\infty} \frac{J_1(Z_m a)}{J_1(Z_m b)} \frac{f_m}{Z_m} \frac{1}{\delta_m^2 - \alpha^2} + \mathrm{e}^{\mathrm{i}\alpha l} \frac{b}{2} \sum_{m=0}^{\infty} \frac{J_0(\xi_m)}{\alpha_m^2 - \alpha^2} \left[g_m - \mathrm{i}\alpha h_m\right]. \tag{32}$$

2.2. Approximate solution of the modified Wiener-Hopf Equation for $kl >> 1$

By using the factorization and the decomposition procedures, together with the Liouville theorem, the modified Wiener-Hopf equation in (32) can be reduced to the following system of Fredholm integral equations of the second kind:

$$\frac{\dot{F}_+(b, \alpha)}{(k+\alpha) R_+(\alpha)} = -\frac{1}{2\pi i} \int_{L^+} \frac{\dot{F}_-(b, \tau) R_-(\tau) Q(\tau) \mathrm{e}^{-\mathrm{i}\tau l}}{(k+\tau)(\tau - \alpha)} \mathrm{d}\tau \tag{33a}$$

$$+\frac{b}{2} \sum_{m=0}^{\infty} \frac{J_0(\xi_m) \left[g_m + \mathrm{i}\alpha_m h_m\right] (k+\alpha_m) R_+(\alpha_m)}{2\alpha_m (\alpha + \alpha_m)} - \frac{\mathrm{i}}{2\pi} \sum_{m=0}^{\infty} \frac{J_1(Z_m a)}{J_1(Z_m b)} \frac{f_m}{Z_m} \frac{k+\delta_m}{\delta_m + \alpha} R_+(\delta_m) \mathrm{e}^{\mathrm{i}\delta_m l},$$

$$\frac{\dot{F}_-(b,\alpha)Q_-(\alpha)}{(k-\alpha)} = \frac{1}{2\pi\mathrm{i}}\int_{L^-}\frac{\dot{F}_+(b,\tau)\mathrm{e}^{\mathrm{i}\tau l}}{(k-\tau)R(\tau)Q_+(\tau)(\tau-\alpha)}\mathrm{d}\tau$$
$$-\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m-\mathrm{i}\alpha_m h_m\right](k+\alpha_m)\mathrm{e}^{\mathrm{i}\alpha_m l}}{2\alpha_m(\alpha-\alpha_m)Q_+(\alpha_m)}+\frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty}\frac{J_1(Z_m a)}{J_1(Z_m b)}\frac{f_m}{Z_m}\frac{k+\delta_m}{\delta_m-\alpha}\frac{1}{Q_+(\delta_m)}, \tag{33b}$$

where the paths of integration L^+ and L^- are depicted in Figure 2. Here, $R_+(\alpha)$, $Q_+(\alpha)$ and $R_-(\alpha) = R_+(-\alpha)$, $Q_-(\alpha) = Q_+(-\alpha)$ are the split functions, regular and free of zeros in the upper ($\Im\mathrm{m}\alpha > \Im\mathrm{m}(-k)$) and lower ($\Im\mathrm{m}\alpha < \Im\mathrm{m}k$) halves of the complex α-plane, respectively, resulting from the Wiener-Hopf factorization of $R(\alpha)$ and $Q(\alpha)$ which are given by (26b) and (26c), in the following form:

$$R(\alpha) = R_+(\alpha)R_-(\alpha), \quad Q(\alpha) = Q_+(\alpha)Q_-(\alpha). \tag{34a,b}$$

The explicit expressions for $R_+(\alpha)$ and $Q_+(\alpha)$ can be obtained by using the results of [4], [11] and [12] as follows:

$$R_+(\alpha) = \left[\pi\mathrm{i}J_1(kb)H_1^{(1)}(kb)\right]^{1/2}\exp\left\{\mathrm{i}\frac{\alpha b}{\pi}\left[1-\gamma+\log(\frac{2\pi}{kb})+\mathrm{i}\frac{\pi}{2}\right]-\mathrm{i}\frac{kb}{2}\right\}$$
$$\times\exp\left\{\frac{K(\alpha)b}{\pi}\log(\frac{\alpha+\mathrm{i}K(\alpha)}{k})+q_1(\alpha)\right\}\prod_{m=1}^{\infty}(1+\frac{\alpha}{\alpha_m})\exp(\frac{\mathrm{i}\alpha b}{m\pi}), \tag{35a}$$

$$Q_+(\alpha) = \left[\frac{H_1^{(1)}(ka)}{\pi H_1^{(1)}(kb)\left[J_1(ka)Y_1(kb)-J_1(kb)Y_1(ka)\right]}\right]^{1/2}$$
$$\times\prod_{m=1}^{\infty}\frac{1}{(1+\alpha/\delta_m)\mathrm{e}^{-\alpha/\delta_m}}\exp\left[\frac{\mathrm{i}k(b-a)}{2}+\frac{K(\alpha)(a-b)}{\pi}\log\frac{\alpha+\mathrm{i}K(\alpha)}{k}+q_2(\alpha)-q_1(\alpha)\right]$$
$$\exp\left\{\frac{\alpha}{\pi\mathrm{i}}(b-a)\left[1-\gamma+\log\left(\frac{2\pi\mathrm{i}}{k(b-a)}\right)\right]\right\}, \tag{35b}$$

where γ is the Euler's constant given by $\gamma = 0{\cdot}57721\ldots$ and $q_{1,2}(\alpha)$ stands for

$$q_1(\alpha) = \frac{1}{\pi}\mathcal{P}\int_0^{\infty}\left[1-\frac{2}{\pi x}\frac{1}{J_1^2(x)+Y_1^2(x)}\right]\log\left(1+\frac{\alpha b}{\left[(kb)^2-x^2\right]^{1/2}}\right)\mathrm{d}x \tag{35c}$$

$$q_2(\alpha) = \frac{1}{\pi}\mathcal{P}\int_0^{\infty}\left[1-\frac{2}{\pi x}\frac{1}{J_1^2(x)+Y_1^2(x)}\right]\log\left(1+\frac{\alpha a}{\left[(ka)^2-x^2\right]^{1/2}}\right)\mathrm{d}x. \tag{35d}$$

In (35c,d), the letter $\mathcal{P}$ denotes the Cauchy principle value at the singularities $x = ka$ and $x = kb$. Note that, when we let $|\alpha| \to \infty$ in their respective regions of regularity, we have

$$R_\pm(\alpha) \sim (\pm\alpha)^{-1/2}, \quad Q_\pm(\alpha) = (\pm\alpha)^{1/2}. \tag{35e,f}$$

For $kl >> 1$, the coupled system of Fredholm integral equations of the second kind in (33a) and (33b), is susceptible to a treatment by iterations.

$$\dot{F}_+(b,\alpha) = \dot{F}_+^{(1)}(b,\alpha) + \dot{F}_+^{(2)}(b,\alpha) + \cdots \tag{36a}$$

$$\dot{F}_-(b,\alpha) = \dot{F}_-^{(1)}(b,\alpha) + \dot{F}_-^{(2)}(b,\alpha) + \cdots \tag{36b}$$

The first iteration gives

$$\begin{aligned}\frac{\dot{F}_+^{(1)}(b,\alpha)}{(k+\alpha)R_+(\alpha)} = {} & \frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m + \mathrm{i}\alpha_m h_m\right](k+\alpha_m)R_+(\alpha_m)}{2\alpha_m(\alpha+\alpha_m)} \\ & -\frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty} S_m \frac{k+\delta_m}{\delta_m+\alpha} R_+(\delta_m)\mathrm{e}^{\mathrm{i}\delta_m l}\end{aligned} \tag{37a}$$

and

$$\begin{aligned}\frac{\dot{F}_-^{(1)}(b,\alpha)Q_-(\alpha)}{(k-\alpha)} = {} & -\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m - \mathrm{i}\alpha_m h_m\right](k+\alpha_m)\mathrm{e}^{\mathrm{i}\alpha_m l}}{2\alpha_m(\alpha-\alpha_m)Q_+(\alpha_m)} \\ & +\frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty} S_m \frac{k+\delta_m}{\delta_m-\alpha} \frac{1}{Q_+(\delta_m)},\end{aligned} \tag{37b}$$

while the second iteration reads

$$\begin{aligned}\frac{\dot{F}_+^{(2)}(b,\alpha)}{(k+\alpha)R_+(\alpha)} = {} & -\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m - \mathrm{i}\alpha_m h_m\right](k+\alpha_m)\mathrm{e}^{\mathrm{i}\alpha_m l}}{2\alpha_m Q_+(\alpha_m)} I_1(\alpha) \\ & +\frac{i}{2\pi}\sum_{m=0}^{\infty} S_m \frac{k+\delta_m}{Q_+(\delta_m)} I_2(\alpha),\end{aligned} \tag{38a}$$

$$\begin{aligned}\frac{\dot{F}_-^{(2)}(b,\alpha)Q_-(\alpha)}{(k-\alpha)} = {} & \frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m + \mathrm{i}\alpha_m h_m\right](k+\alpha_m)R_+(\alpha_m)}{2\alpha_m} I_3(\alpha) \\ & -\frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty} S_m (k+\delta_m) R_+(\delta_m)\mathrm{e}^{\mathrm{i}\delta_m l} I_4(\alpha),\end{aligned} \tag{38b}$$

with

$$I_1(\alpha) = -\frac{1}{2\pi\mathrm{i}}\int_{L^+}\frac{R_-(\tau)Q(\tau)(k-\tau)\mathrm{e}^{-\mathrm{i}\tau l}}{(k+\tau)(\tau-\alpha_m)Q_-(\tau)}\frac{\mathrm{d}\tau}{(\tau-\alpha)}, \tag{39a}$$

$$I_2(\alpha) = -\frac{1}{2\pi\mathrm{i}}\int_{L^+}\frac{R_-(\tau)Q(\tau)(k-\tau)\mathrm{e}^{-\mathrm{i}\tau l}}{Q_-(\tau)(\delta_m-\tau)(k+\tau)}\frac{\mathrm{d}\tau}{(\tau-\alpha)}, \tag{39b}$$

$$I_3(\alpha) = \frac{1}{2\pi\mathrm{i}}\int_{L^-}\frac{R_+(\tau)(k+\tau)\mathrm{e}^{\mathrm{i}\tau l}}{(k-\tau)R(\tau)Q_+(\tau)(\tau+\alpha_m)}\frac{\mathrm{d}\tau}{(\tau-\alpha)}, \tag{39c}$$

$$I_4(\alpha) = \frac{1}{2\pi\mathrm{i}}\int_{L^-}\frac{R_+(\tau)(k+\tau)\mathrm{e}^{\mathrm{i}\tau l}}{(k-\tau)R(\tau)Q_+(\tau)(\delta_m+\tau)(\tau-\alpha)}\mathrm{d}\tau, \tag{39d}$$

and

$$S_m = \frac{J_1(Z_m a)}{J_1(Z_m b)} \frac{f_m}{Z_m}. \tag{39e}$$

Consider first the asymptotic evaluation of $I_1(\alpha)$ for $kl \gg 1$. According to Jordan's Lemma, the integration line L^+ can be deformed onto the branch-cut $C_1 + C_2 + C_\epsilon$ through the branch point $\tau = -k$ (see Figure 2). During this deformation one crosses the poles occurring at the zeros of $J_1(Ka)$, lying in the lower half-plane, namely: $\tau = -\alpha_m$ where α_m is given by (22b). The residue contribution of these poles and the contribution from C_ϵ are

$$I_{1_{\mathrm{res}}}(\alpha) = \frac{2k\pi b^2}{(a^2 - b^2)} \frac{R_+(k) \mathrm{e}^{\mathrm{i}kl}}{Q_+(k)(k+\alpha)(k+\alpha_m)} + \sum_{n=0}^{\infty} \frac{(k+\delta_n) R_+(\delta_n) H_1^{(1)}(Z_n a) \mathrm{e}^{\mathrm{i}\delta_n l}}{(k-\delta_n) H_1^{(1)}(Z_n b) Q_+(\delta_n)(\delta_n + \alpha_m)(\delta_n + \alpha) \dot{M}(-\delta_n)}. \tag{40a}$$

If we denote the branch-cut contribution to (39a) by $I_{1_{\mathrm{bc}}}(\alpha)$, we can now write

$$I_1(\alpha) = I_{1_{\mathrm{res}}}(\alpha) + I_{1_{\mathrm{bc}}}(\alpha). \tag{40b}$$

Consider now the branch-cut integral $I_{1_{\mathrm{bc}}}(\alpha)$ which can be rearranged as follows:

$$I_{1_{\mathrm{bc}}}(\alpha) = -\frac{1}{2\pi \mathrm{i}} \left[\int_{C_1} \frac{R_-(\tau) Q(\tau)(k-\tau) \mathrm{e}^{-\mathrm{i}\tau l}}{(\tau - \alpha_m) Q_-(\tau)(k+\tau)} \frac{\mathrm{d}\tau}{(\tau - \alpha)} + \int_{C_2} \frac{R_-(\tau) Q(\tau)(k-\tau) \mathrm{e}^{-\mathrm{i}\tau l}}{(\tau - \alpha_m) Q_-(\tau)(k+\tau)} \frac{\mathrm{d}\tau}{(\tau - \alpha)} \right]. \tag{40c}$$

Using the properties

$$J_1(\mathrm{e}^{\mathrm{i}\pi} Ka) = -J_1(Ka), \quad H_1^{(1)}(\mathrm{e}^{\mathrm{i}\pi} Ka) = H_1^{(2)}(Ka) = J_1(Ka) - \mathrm{i}Y_1(Ka), \tag{41a,b}$$

and making the following substitution

$$k + \tau = t\mathrm{e}^{-\mathrm{i}\pi/2}, \quad t > 0, \tag{42}$$

the integral in (40c) can be reduced to the following one written over $\mathbb{R}^+$

$$I_{1_{\mathrm{bc}}}(\alpha) = -\frac{1}{\pi^2} \int_0^\infty U(t) \frac{\mathrm{e}^{-\tau l}}{(k + \mathrm{i}t + \alpha)} \mathrm{d}t, \tag{43a}$$

with

$$U(t) = -\frac{R_+(k+\mathrm{i}t) \mathrm{e}^{\mathrm{i}kl}}{t(J_1^2(Kb) + Y_1^2(Kb) Q_+(k+\mathrm{i}t)(k+\mathrm{i}t+\alpha_m)}. \tag{43b}$$

If kl is large, the main contribution to the integral in (37a) comes from the end point $t = 0$. Hence, after replacing Bessel's functions by their following asymptotic expression valid for small arguments

$$J_1^2(z) + Y_1^2(z) \sim \frac{4}{\pi^2 z^2}, \quad |z| \to 0, \tag{44}$$

we can take $U(t)$ out from the integral by assigning its value at $t = 0$. The resultant integral can be evaluated easily; as a result we obtain

$$I_{1_{bc}}(\alpha) \approx -(kb)^2 \frac{R_+(k)}{Q_+(k)(k+\alpha_m)} \mathrm{e}^{\mathrm{i}kl} W_{-1/2}(-\mathrm{i}l(\alpha+k)), \tag{45}$$

with

$$W_{-1/2}(\xi) = \int_0^\infty \frac{\exp(-u)}{u+\xi} \mathrm{d}u. \tag{46a}$$

The function $W_{-1/2}(\xi)$ is related to the Whittaker function $W_{-1/2,0}(\xi)$ [13, Chapter 16] by the relation

$$W_{-1/2}(\xi) = \exp(\xi/2)\, \xi^{-1/2} W_{-1/2,0}(\xi). \tag{46b}$$

Finally we obtain

$$\begin{aligned} I_1(\alpha) = &-(kb)^2 \frac{R_+(k)}{Q_+(k)(k+\alpha_m)} \mathrm{e}^{\mathrm{i}kl} W_{-1/2}(-\mathrm{i}l(\alpha+k)) \\ &+ \sum_{n=0}^{\infty} \frac{(k+\delta_n) R_+(\delta_n) H_1^{(1)}(Z_n a) \mathrm{e}^{\mathrm{i}\delta_n l}}{(k-\delta_n) H_1^{(1)}(Z_n b) Q_+(\delta_n)(\delta_n+\alpha_m)(\delta_n+\alpha)\dot{M}(-\delta_n)} \\ &+ \frac{2k\pi b^2}{(a^2-b^2)} \frac{R_+(k)\mathrm{e}^{\mathrm{i}kl}}{Q_+(k)(k+\alpha)(k+\alpha_m)}. \end{aligned} \tag{47a}$$

By proceeding similarly, we get the following approximate expressions for $I_2(\alpha)$, $I_3(\alpha)$ and $I_4(\alpha)$ valid for $kl \gg 1$. The result can be written as

$$\begin{aligned} I_2(\alpha) = &(kb)^2 \frac{R_+(k)}{Q_+(k)} \frac{\mathrm{e}^{\mathrm{i}kl}}{(k+\delta_n)} W_{-1/2}(-\mathrm{i}l(\alpha+k)) \\ &- \sum_{n=0}^{\infty} \frac{(k+\delta_n) R_+(\delta_n) H_1^{(1)}(Z_n a) \mathrm{e}^{\mathrm{i}\delta_n l}}{(\delta_m+\delta_n)(k-\delta_n) H_1^{(1)}(Z_n b) Q_+(\delta_n)(\delta_n+\alpha)\dot{M}(-\delta_n)} \\ &- \frac{2k\pi b^2}{(a^2-b^2)} \frac{R_+(k)\mathrm{e}^{\mathrm{i}kl}}{Q_+(k)(k+\alpha)(k+\delta_m)}. \end{aligned} \tag{47b}$$

$$I_3(\alpha) \approx (kb)^2 \frac{R_+(k)}{Q_+(k)(\alpha_m+k)} \mathrm{e}^{\mathrm{i}kl} W_{-1/2}(\mathrm{i}l(\alpha-k)) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{R_+(\alpha_n)(k+\alpha_n)^2 \mathrm{e}^{\mathrm{i}\alpha_n l}}{\alpha_n(\alpha-\alpha_n)(\alpha_n+\alpha_m) Q_+(\alpha_n)}, \tag{47c}$$

$$I_4(\alpha) = (kb)^2 \frac{R_+(k)}{Q_+(k)(\delta_m+k)} \mathrm{e}^{\mathrm{i}kl} W_{-1/2}(\mathrm{i}l(\alpha-k) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{R_+(\alpha_n)(k+\alpha_n)^2 \mathrm{e}^{\mathrm{i}\alpha_n l}}{\alpha_n(\alpha-\alpha_n)(\alpha_n+\delta_m) Q_+(\alpha_n)}. \tag{47d}$$

Now, the approximate solution of the modified Wiener-Hopf equation reads:

$$\frac{\dot{F}_+(b,\alpha)}{(k+\alpha)R_+(\alpha)} \simeq \frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m + i\alpha_m h_m\right](k+\alpha_m)R_+(\alpha_m)}{2\alpha_m(\alpha+\alpha_m)}$$
$$-\frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty} S_m \frac{k+\delta_m}{\delta_m+\alpha} R_+(\delta_m)\mathrm{e}^{\mathrm{i}\delta_m l}$$
$$-\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m - i\alpha_m h_m\right](k+\alpha_m)\mathrm{e}^{\mathrm{i}\alpha_m l}}{2\alpha_m Q_+(\alpha_m)} I_1(\alpha)$$
$$+\frac{i}{2\pi}\sum_{m=0}^{\infty} S_m \frac{k+\delta_m}{Q_+(\delta_m)} I_2(\alpha), \tag{48a}$$

$$\frac{\dot{F}_-(b,\alpha)Q_-(\alpha)}{(k-\alpha)} \simeq -\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m - \mathrm{i}\alpha_m h_m\right](k+\alpha_m)\mathrm{e}^{\mathrm{i}\alpha_m l}}{2\alpha_m(\alpha-\alpha_m)Q_+(\alpha_m)}$$
$$+\frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty} S_m \frac{k+\delta_m}{\delta_m-\alpha}\frac{1}{Q_+(\delta_m)}$$
$$+\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m + \mathrm{i}\alpha_m h_m\right](k+\alpha_m)R_+(\alpha_m)}{2\alpha_m} I_3(\alpha)$$
$$-\frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty} S_m(k+\delta_m)R_+(\delta_m)\mathrm{e}^{\mathrm{i}\delta_m l} I_4(\alpha). \tag{48b}$$

2.3. DETERMINATION OF THE EXPANSION COEFFICIENTS

The field in the cavity can be expressed in terms of the waveguide normal modes as follows

$$u_3(\rho,z) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-\mathrm{i}\beta_n z} J_0(\xi_n \frac{\rho}{a}), \tag{49a}$$

with

$$\beta_n = \sqrt{k^2 - \frac{\xi_n^2}{a^2}}, \quad n = 0, 1, 2 \dots . \tag{49b}$$

Here ξ_n's are the roots of the characteristic equation

$$J_1(\xi_n) = 0, \quad n = 0, 1, 2 \dots . \tag{49c}$$

Similarly, in the region $0 < \rho < b$, $0 < z < l$, $u_4\,(\rho, z)$ can be expressed in terms of the following normal waveguide modes

$$u_4(\rho,z) = \sum_{n=0}^{\infty}\left(p_n \mathrm{e}^{\mathrm{i}\alpha_n z} + q_n \mathrm{e}^{-\mathrm{i}\alpha_n z}\right) J_0(\xi_n \frac{\rho}{b}). \tag{50a}$$

Now, from the continuity relations (4j–m) we write

$$\frac{\partial}{\partial z}u_4(\rho,0)=\begin{cases}\dfrac{\partial}{\partial z}u_3(\rho,0)+\mathrm{i}k, & \rho\in(0,a)\\ 0, & \rho\in(a,b)\end{cases}, \tag{51a}$$

and

$$u_4(\rho,0)=u_3(\rho,0)+1; \quad \rho\in(0,a)\,, \tag{51b}$$

$$\frac{\partial u_4}{\partial z}(\rho,l)=g(\rho)=\sum_{m=0}^{\infty}g_m J_0(\xi_m\frac{\rho}{b}); \quad \rho\in(0,b), \tag{51c}$$

$$u_4(\rho,l)=h(\rho)=\sum_{m=0}^{\infty}h_m J_0(\xi_m\frac{\rho}{b}); \quad \rho\in(0,b). \tag{51d}$$

Inserting the series expansions of $g\,(\rho)$ and $h\,(\rho)$ given in (28b) and (28c) in (51c) and (51d), respectively, and using (49a) and (50a), we get:

$$-\sum_{n=0}^{\infty}i\alpha_n\left[p_n-q_n\right]J_0(\xi_n\frac{\rho}{b})=\begin{cases}\displaystyle\sum_{m=0}^{\infty}i\beta_m c_m J_0(\xi_m\frac{\rho}{a})-\mathrm{i}k, & \rho\in(0,a)\\ 0, \quad \rho\in(a,b)\end{cases}, \tag{52a}$$

$$\sum_{n=0}^{\infty}\left[p_n+q_n\right]J_0(\xi_n\frac{\rho}{b})=\sum_{m=0}^{\infty}c_m J_0(\xi_m\frac{\rho}{a})+1, \quad \rho\in(0,a)\,, \tag{52b}$$

$$\sum_{n=0}^{\infty}i\alpha_n\left[p_n\mathrm{e}^{\mathrm{i}\alpha_n l}-q_n\mathrm{e}^{-\mathrm{i}\alpha_n l}\right]J_0(\xi_n\frac{\rho}{b})=\sum_{m=0}^{\infty}g_m J_0(\xi_m\frac{\rho}{b}), \tag{52c}$$

and

$$\sum_{n=0}^{\infty}\left[p_n\mathrm{e}^{\mathrm{i}\alpha_n l}+q_n\mathrm{e}^{-\mathrm{i}\alpha_n l}\right]J_0(\xi_n\frac{\rho}{b})=\sum_{m=0}^{\infty}h_m J_0(\xi_m\frac{\rho}{b}). \tag{52d}$$

Multiplying both sides of (52a) and (52b) by $\rho J_0(\xi_l\frac{\rho}{b})$ and by $J_0(\xi_l\frac{\rho}{a})$, respectively, and integrating from 0 to b and from 0 to a, respectively, we obtain the following system of linear algebraic equations:

$$\alpha_0(p_0-q_0)\frac{b^2}{2}=\frac{ka^2}{2}(1-c_0), \qquad n=0, \tag{53a}$$

$$\alpha_n(p_n-q_n)\frac{b^2}{2}J_0^2(\xi_n)=-\frac{a}{b}\sum_{m=0}^{\infty}\beta_m c_m\frac{\xi_n\left[J_0(\xi_m)J_1(\xi_n\frac{a}{b})\right]}{(\xi_n/b)^2-(\xi_m/a)^2}+\frac{kba}{\xi_n}J_1(\xi_n\frac{a}{b}), \quad n=1,2,\ldots, \tag{53b}$$

$$c_0=(p_0+q_0)+\sum_{n=1}^{\infty}(p_n+q_n)\frac{2b}{a\xi_n}J_1(\xi_n\frac{a}{b})-1, \quad m=0, \tag{53c}$$

$$c_m = \frac{2}{abJ_0(\xi_m)}\sum_{n=0}^{\infty}(p_n+q_n)\frac{\xi_n}{(\xi_n/b)^2-(\xi_m/a)^2}J_1(\xi_n\frac{a}{b}), \quad m=1,2,\ldots, \tag{53d}$$

$$g_m = \mathrm{i}\alpha_m\left[p_m\mathrm{e}^{\mathrm{i}\alpha_m l} - q_m\mathrm{e}^{-\mathrm{i}\alpha_m l}\right], \quad m=0,1,2,\cdots, \tag{53e}$$

$$h_m = p_m\mathrm{e}^{\mathrm{i}\alpha_m l} + q_m e^{-\mathrm{i}\alpha_m l}, \quad m=0,1,2,\cdots. \tag{53f}$$

This system of equations can be rearranged as

$$g_m - \mathrm{i}\alpha_m h_m = -2\mathrm{i}\alpha_m q_m\mathrm{e}^{-\mathrm{i}\alpha_m l}, \quad m=0,1,2,\cdots, \tag{54a}$$

$$g_m + \mathrm{i}\alpha_m h_m = 2\mathrm{i}\alpha_m p_m\mathrm{e}^{\mathrm{i}\alpha_m l}, \quad m=0,1,2,\cdots, \tag{54b}$$

$$k(a^2+b^2)p_0 - k(b^2-a^2)q_0 + 2kab\sum_{n=1}^{\infty}(p_n+q_n)\frac{J_1(\xi_n\frac{a}{b})}{\xi_n} = 2ka^2 \quad r=0, \tag{54c}$$

$$\begin{aligned}&\alpha_r(p_r-q_r)\frac{b^2}{2}J_0^2(\xi_r) + \frac{2}{b^2}\sum_{m=1}^{\infty}(p_m+q_m)\xi_m J_1(\xi_m\frac{a}{b})\\ &\quad\sum_{n=1}^{\infty}\frac{\beta_n\xi_r J_1(\xi_r\frac{a}{b})}{\left[(\xi_r/b)^2-(\xi_n/a)^2\right]\left[(\xi_m/b)^2-(\xi_n/a)^2\right]}\\ &\quad+\frac{kab}{\xi_r}J_1(\xi_r\frac{a}{b})(p_0+q_0) + 2kb^2\sum_{m=1}^{\infty}(p_m+q_m)\frac{J_1(\xi_m\frac{a}{b})J_1(\xi_r\frac{a}{b})}{\xi_m\xi_r} = \frac{2kba}{\xi_r}J_1(\xi_r\frac{a}{b}),\\ &\qquad r=1,2,3,\ldots.\end{aligned} \tag{54d}$$

To obtain an approximate value for $\dot{F}_+(a,\alpha)$ and $\dot{F}_-(a,\alpha)$, we substitute $\alpha = k, \alpha_1, \alpha_2, \ldots, \alpha_N$ in (48a) and $\alpha = -\delta_1, -\delta_2, \ldots, -\delta_N$ in (48b). These equations, together with (54c) and (54d), result in $3(N+1)$ equations for $3(N+1)$ unknowns. The solution of these simultaneous equations yields approximate solutions for $\dot{F}_+(b,k)$, $\dot{F}_+(b,\alpha_1)$, $\dot{F}_+(b,\alpha_2), \ldots$ and $\dot{F}_-(b,-\delta_1)$, $\dot{F}_-(b,-\delta_2), \ldots$. By using (31a–b) we obtain

$$\begin{aligned}-\frac{b}{2}\frac{J_0(\xi_r)(g_r-\mathrm{i}\alpha_r h_r)}{2(k+\alpha_r)R_+(\alpha_r)} &= \frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)(k+\alpha_m)}{2\alpha_m}\left\{\frac{(g_m+\mathrm{i}\alpha_m h_m)R_+(\alpha_m)}{\alpha_r+\alpha_m}\right.\\ &\left.-\frac{(g_m-\mathrm{i}\alpha_m h_m)\mathrm{e}^{\mathrm{i}\alpha_m l}}{Q_+(\alpha_m)}A_{rm}\right\} - \frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty}S_m\,(k+\delta_m)\times\left\{\frac{R_+(\delta_m)}{\delta_m+\alpha_r}\mathrm{e}^{\mathrm{i}\delta_m l} - \frac{B_{rm}}{Q_+(\delta_m)}\right\}\end{aligned} \tag{55a}$$

and

$$\begin{aligned}\frac{\mathrm{i}\delta_r}{\pi Z_r}\frac{J_1^2(Z_rb)-J_1^2(Z_ra)}{J_1(Z_ra)J_1(Z_rb)}\frac{Q_+(\delta_r)f_r}{(k+\delta_r)} &= \frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)(k+\alpha_m)}{2\alpha_m}\{(g_m+\mathrm{i}\alpha_m h_m)R_+(\alpha_m)C_{rm}\\ &+\left.\frac{(g_m-\mathrm{i}\alpha_m h_m)\mathrm{e}^{\mathrm{i}\alpha_m l}}{(\alpha_m+\delta_r)Q_+(\alpha_m)}\right\} - \frac{\mathrm{i}}{2\pi}\sum_{m=0}^{\infty}S_m\,(k+\delta_m)\left\{R_+(\delta_m)\mathrm{e}^{\mathrm{i}\delta_m l} - \frac{1}{Q_+(\delta_m)(\delta_m+\delta_r)}\right\},\end{aligned} \tag{55b}$$

with

$$A_{rm} = \left\{ -(kb)^2 \frac{R_+(k)}{Q_+(k)} \frac{e^{ikl}}{(k+\alpha_m)} W_{-1/2}(-il(\alpha_r + k)) + \frac{2k\pi b^2}{(a^2-b^2)} \frac{R_+(k)e^{ikl}}{Q_+(k)(k+\alpha_r)(k+\alpha_m)} \right.$$

$$\left. + \sum_{n=0}^{\infty} \frac{(k+\delta_n)R_+(\delta_n)H_1^{(1)}(Z_n a)e^{i\delta_n l}}{(k-\delta_n)H_1^{(1)}(Z_n b)Q_+(\delta_n)(\delta_n+\alpha_m)(\delta_n+\alpha_r)\dot{M}(-\delta_n)} \right\}, \tag{55c}$$

$$B_{rm} = (kb)^2 \frac{R_+(k)}{Q_+(k)} \frac{e^{ikl}}{(k+\delta_m)} W_{-1/2}(-il(\alpha_r + k)) - \frac{2k\pi b^2}{(a^2-b^2)} \frac{R_+(k)e^{ikl}}{Q_+(k)(k+\alpha_r)(k+\delta_m)}$$

$$- \sum_{n=0}^{\infty} \frac{(k+\delta_n)R_+(\delta_n)H_1^{(1)}(Z_n a)e^{i\delta_n l}}{(\delta_m+\delta_n)(k-\delta_n)H_1^{(1)}(Z_n b)Q_+(\delta_n)(\delta_n+\alpha_r)\dot{M}(-\delta_n)}, \tag{55d}$$

$$C_{rm} = (kb)^2 \frac{R_+(k)}{Q_+(k)(\alpha_m+k)} e^{ikl} W_{-1/2}(-il(\delta_r + k)) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{R_+(\alpha_n)(k+\alpha_n)^2 e^{i\alpha_n l}}{\alpha_n(\delta_r+\alpha_n)(\alpha_n+\alpha_m)Q_+(\alpha_n)}, \tag{55e}$$

$$D_{rm} = (kb)^2 \frac{R_+(k)}{Q_+(k)(\delta_m+k)} e^{ikl} W_{-1/2}(-il(\delta_r + k)) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{R_+(\alpha_n)(k+\alpha_n)^2 e^{i\alpha_n l}}{\alpha_n(\delta_r+\alpha_n)(\alpha_n+\delta_m)Q_+(\alpha_n)}. \tag{55f}$$

By substituting (54a,b) in (55a,b) and also considering (54c–d), we can easily obtain the three infinite systems of linear algebraic equations with coefficients p_n, q_n and f_n.

3. The radiated far-field and computational results

The radiated field in the region $\rho > b$ can be obtained by taking the inverse Fourier transform of $F(\rho, \alpha)$. From (7a) and (10) we obtain

$$u_1(\rho, z) = -\frac{1}{2\pi} \int_L \frac{H_0^{(1)}(K\rho)}{K(\alpha)H_1^{(1)}(Kb)} [\dot{F}_-(b,\alpha) + e^{i\alpha l}\dot{F}_+(b,\alpha)] e^{-i\alpha z} d\alpha, \tag{56}$$

where L is a straight line parallel to the real α-axis, lying in the strip $\mathfrak{Im}(-k) < \mathfrak{Im}(\alpha) < \mathfrak{Im}(k)$. Utilizing the asymptotic expansion of $H_0^{(1)}(K\rho)$ as $k\rho \to \infty$

$$H_0^{(1)}(K\rho) = \sqrt{\frac{2}{\pi K\rho}} e^{i(K\rho - \pi/4)} \tag{57}$$

we observe that the asymptotic evaluation of the integral in (56), using the saddle point technique, yields for the diffracted field for $k\sqrt{\rho^2+z^2} >> kl$,

$$u_1(\rho, z) \approx \frac{i}{\pi} \left\{ \frac{e^{ikr_1}}{kr_1} \frac{\dot{F}_+(b, -k\cos\theta_1)}{\sin\theta_1 H_1^{(1)}(kb\sin\theta_1)} + \frac{e^{ikr_2}}{kr_2} \frac{\dot{F}_-(b, -k\cos\theta_2)}{\sin\theta_2 H_1^{(1)}(kb\sin\theta_2)} \right\}, \tag{58a}$$

where $\dot{F}_+(a,\alpha)$ and $\dot{F}_-(a,\alpha)$ are given by (48a) and (48b), respectively. Here r_1, θ_1, and r_2, θ_2 are the spherical coordinates defined by

$$\rho = r_1 \sin\theta_1, \quad z = r_1 \cos\theta_1 \tag{58b}$$

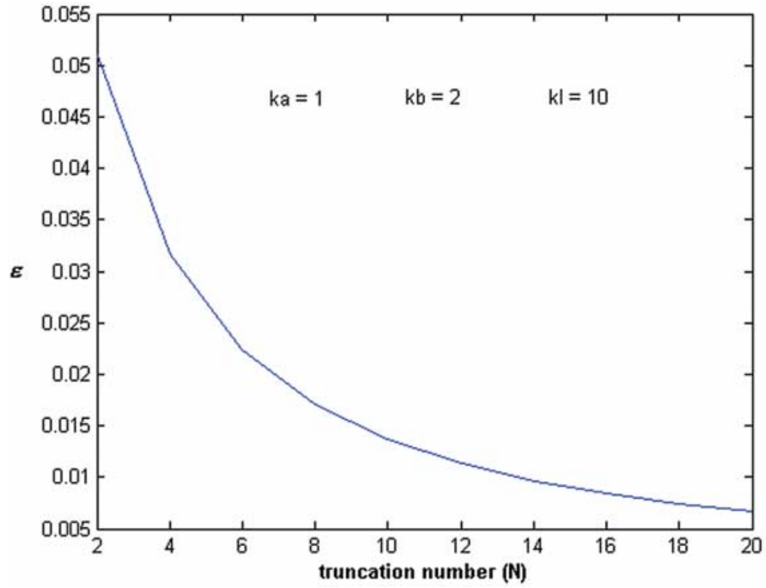

Figure 4. Absolute error in the fulfilment of the continuity relation *vs.* the truncation number N

and

$$\rho = r_2 \sin\theta_2, \quad z - l = r_2 \cos\theta_2. \tag{58c}$$

In the far-field region we have

$$\theta_1 \approx \theta_2 \tag{59a}$$

$$r_2 = \sqrt{r_1^2 + l^2 - 2r_1 l \cos\theta_1} \approx \begin{cases} r_1 - l\cos\theta_1, & \text{for the phase term} \\ r_1, & \text{for the amplitude term} \end{cases} \tag{59b}$$

and (58) reduces to

$$u_1(\rho, z) \approx \frac{\mathrm{i}}{\pi}\left\{\frac{\dot{F}_+(b, -k\cos\theta_1) + \mathrm{e}^{-\mathrm{i}kl\cos\theta_1}\,\dot{F}_-(b, -k\cos\theta_1)}{\sin\theta_1 H_1^{(1)}(kb\sin\theta_1)}\right\}\frac{\mathrm{e}^{\mathrm{i}kr_1}}{kr_1}. \tag{60}$$

From (31e) and (54a) we can see that f_m and q_m decay exponentially with m, so that the infinite algebraic systems converge very rapidly. Thus, they can be solved by truncating the infinite matrix and numerically inverting the resulting finite system. The value of the truncation number N is increased until the final physical quantities, such as the amplitude of the radiated field or the reflection coefficients, become insensitive up to the desired digit after the decimal point.

From Table-1 it is seen that the relative errors made in calculating the reflection coefficient $|q_0|$ by choosing $N = 2$ and $N = 20$ are 0·5% for $k = 1$, 1·1% for $k = 2$ and 1·4% for $k = 3$, respectively. Thus, for higher frequencies the error can be reduced by increasing the truncation number N.

Another effective check on the analysis can be made by showing numerically that the continuity relation in (52b) is satisfied. Figure 4 displays the variation of the absolute error in the

Table 1. Reflection coefficient $|q_0|$ versus the truncation number N for different of k.

N	$ka = 0{\cdot}5, kb = 1, kl = 5$	$ka = 1, kb = 2, kl = 10$	$ka = 1{\cdot}5, kb = 3, kl = 15$
2	0·296464	0·203286	0·151387
4	0·295555	0·201966	0·150149
6	0·295277	0·201542	0·149749
8	0·295152	0·201346	0·149561
10	0·295085	0·201237	0·149455
12	0·295022	0·201167	0·149388
14	0·294960	0·201118	0·149342
16	0·294912	0·201084	0·149309
18	0·294875	0·201058	0·149285
20	0·294843	0·201038	0·149266

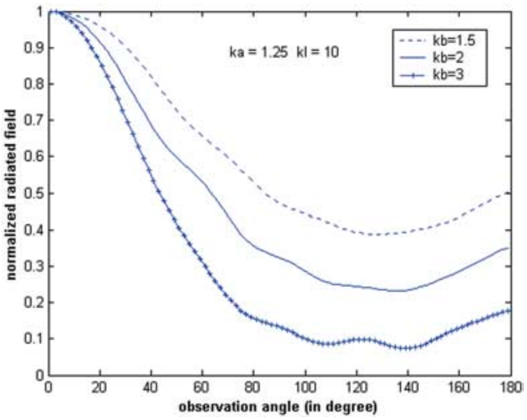

Figure 5a. Normalized field amplitude *vs.* the observation angle for different values of *kb* with *ka* fixed.

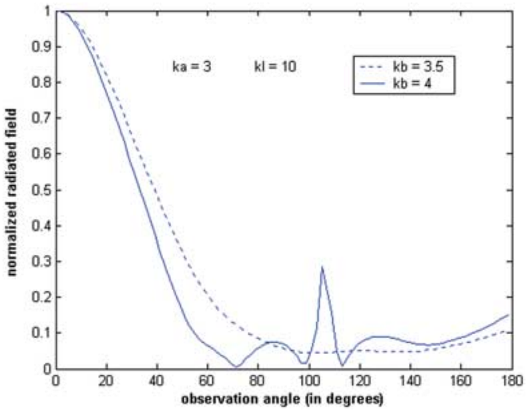

Figure 5b. Normalized field amplitude *vs.* the observation angle for different values *kb* with *ka* fixed.

fulfillment of the continuity condition in (52b) at $\rho = 0$, *i.e.*, $\varepsilon = \left|\sum_{n=0}^{\infty} p_n + q_n - c_n - 1\right|$ *vs.* the truncation number N. The absolute error is less than 1% for $N \geq 14$

Figures 5a,b show the variation of the normalized diffracted field amplitude $|u_1(r_1, \theta_1) / u_1(r_1, 0)|$ *vs.* the observation angle θ_1, for different values of $k(b-a)$ when ka is fixed.

Note that the directivity of the stepped waveguide increases with increasing values of the step height $k(b-a)$. As can be shown from Figure 5b, when kb exceeds 3·83 (the second zero of J_1), more than one duct mode can be cut-on. For instance, choosing $kb = 4$ a lobed radiation pattern with a null near $\theta_1 = 73°$ is observed.

From Figure 6, one can see that, when kb is fixed, the amplitude of the radiated field increases with increasing values of ka.

Figures 7 and 8 depict the variations of the reflection coefficients $|c_0|$ at $z = 0$ and $|q_0|$ at $z = l$ with ka for different values of kl and with kl for different values of kb, respectively. We observe that, when ka and kb increase, the moduli of the reflection coefficients $|c_0|$ and

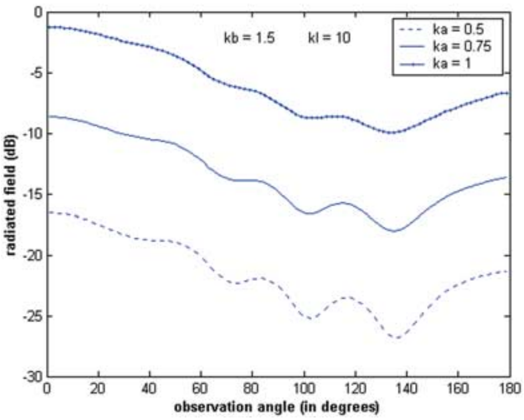

Figure 6. $20\log_{10}|u_1(r_1,\theta_1)\,kr_1|$ *vs.* the observation angle for different values of ka while kb and kl are fixed.

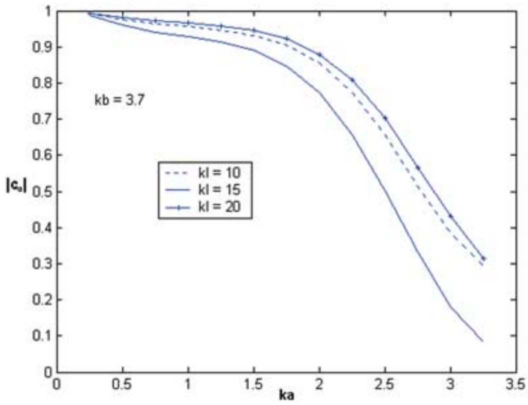

Figure 7. Amplitude of the reflection coefficient $|c_0|$ *vs.* ka, for different values of kl while kb is fixed.

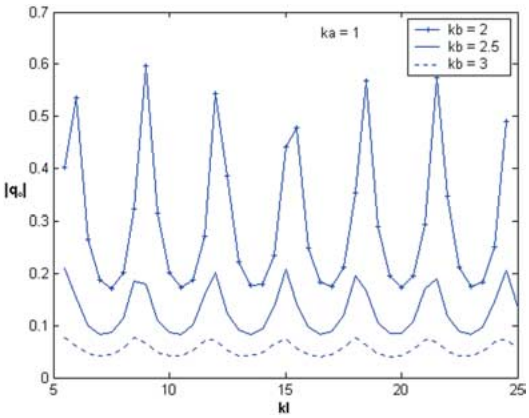

Figure 8. Amplitude of the reflection coefficient $|q_0|$ *vs.* kl, for different values of kb.

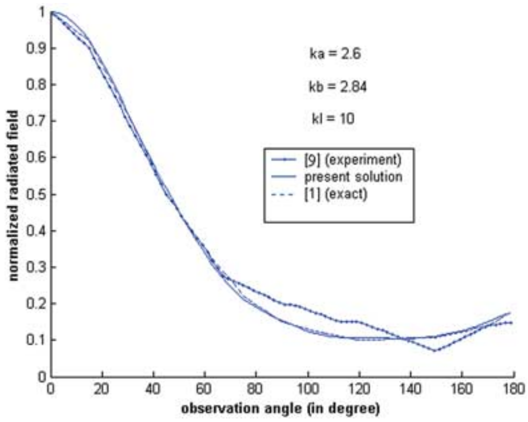

Figure 9. Comparison with the exact solution related to a semi-infinite cylindrical pipe.

$|q_0|$ decrease and the transmission efficiency increases as expected. Notice that $|c_0|$ and $|q_0|$ exhibit an oscillatory behavior when kl varies.

Finally, Figure 9 displays the amplitude of the normalized radiated field obtained in the present work for $ka = 2{\cdot}6$, $kb = 2{\cdot}84$, $kl = 10$, the theoretical results related to a rigid cylindrical pipe of radius $kb = 2{\cdot}84$, derived by [1] and the experimental data provided by [9]. Since our result is obtained under the assumption $a < b$, it is not possible to reduce it to the case of a semi-infinite cylindrical pipe by letting $a \to b$. However, we can see that the results obtained in this work approache the exact solution for $k\,(b - a) = 0{\cdot}24 < 1$ and fits quite well with the experimental data.

4. Concluding remarks

The radiation of sound from a stepped circular cylindrical waveguide has been investigated by using the mode-matching method in conjunction with the Wiener-Hopf technique. The problem was first reduced to a system of Fredholm integral equations of the second kind and then solved approximately by iteration for large kl. The solution involves three systems of linear algebraic equations involving three sets of infinitely many unknown expansion coefficients. A numerical solution to these systems has been obtained for various values of the stepped waveguide parameters, such as waveguide radius a, aperture radius b, and horn length l. In the case where the step height is small, the results obtained in this paper have been compared with the exact solution related to a semi-infinite cylindrical pipe and the agreement was found to be very satisfactory. Furthermore, it has been shown numerically that the error in the fulfilment of the continuity relation (52b) is satisfiactory. This can be considered as a good check for the reliability of the analysis made in this paper.

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References

1. H. Levine and J. Schwinger, On the sound radiation of sound from an unflanged circular pipe. *J. Phys. Rev.* 73 (1948) 383–406.
2. Y. Ando, On the sound radiation from semi-infinite circular pipe of certain wall thickness. *Acustica* 22 (1969/70) 219–225.
3. A. D. Rawlins, Radiation of sound from an unflanged rigid cylindrical duct with an acoustically absorbing internal surface. *Proc. R. Soc. London* A-361 (1978) 65–91.
4. A. Büyükaksoy and B. Polat, Diffraction of acoustic waves by semi-infinite cylindrical impedance pipe of certain wall thickness. *J. Eng. Math*. 33 (1998) 333–352.
5. A. H. Nayfeh and D.P. Telionis, Acoustic propagation in ducts with varying cross-sections. *J. Acoust. Soc. Am.* 54 (1973) 1654–1661.
6. S. W. Rienstra, Sound transmission in slowly varying circular and annular lined ducts with flow. *J. Fluid. Mech* 380 (1999) 279–296.
7. B. Nilsson and O. Brander, The propagation of sound in cylindrical ducts with mean flow and bulk reacting lining III. Step discontinuities. *IMA J. Appl. Math* 27 (1981) 105–131.
8. F. Birbir, A. Büyükaksoy and V. P. Chumachenko, Wiener-Hopf analysis of the two-dimensional box-like horn radiator. *Int. J. Eng. Sci*. 40 (2002) 51–66.
9. Y. Ando, Experimental studies of the pressure directivity and the acoustic center of the circular pipe horn loud speaker. *Acustica* 20 (1968) 366–369.
10. I.H. Sneddon, *The Use of Integral Transforms*. New York: Mc Graw Hill (1972).
11. A. D. Rawlins, A bifurcated circular waveguide problem. *IMA J. Appl. Math*. 54 (1995) 59–81.
12. S-W Lee, V. Jamnejad and R. Mittra, Near field of scattering by a hollow semi-infinite cylinder and its application to sensor booms. *IEEE Trans Antennas and Propagat*. AP-21 (1973) 182–188.
13. E. T. Whittaker and G. N. Watson, Modern Analysis. Cambridge: CLIP (1958) 608pp.